Dynamics of quantum and classical correlations in the three-qubit multiphoton Tavis-Cummings model with a vacuum resonator field

Bashkirov E. K., Bagrov A. R. [a]

Samara University

alexander.bagrov00@mail.ru

[a] Department of General and Theoretical Physics

1. The model and its exact solution

We consider three identical two-level natural or artificial atoms (qubits) A, B, and C, which are trapped in a single-mode infinite-Q resonator and resonantly interact with the mode of the vacuum field via multiphoton transitions. We assume that all qubit-cavity couplings are equal, i.e., $\gamma_A = \gamma_B = \gamma_C \equiv \gamma$. The configuration of the model is shown in Fig. 1.

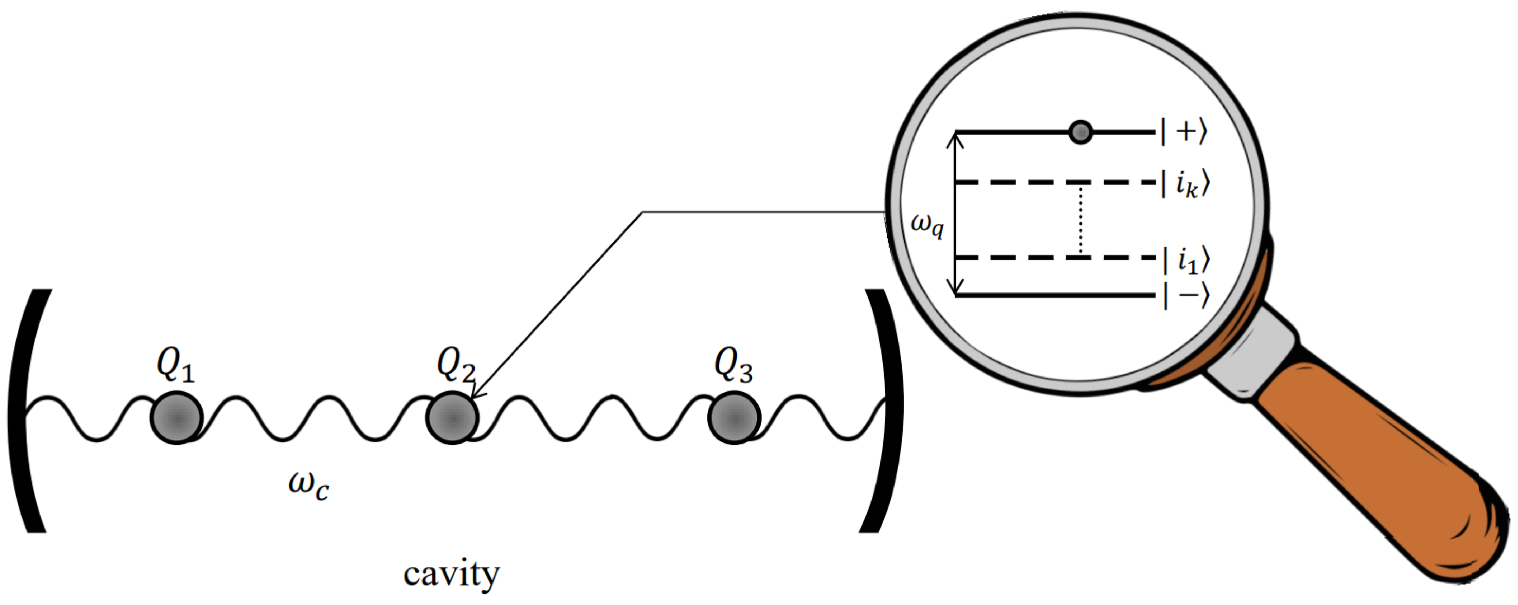

Figure 1: *Configuration of the model under study. Here, ω_c is the frequency of the resonator, $\omega_a = m\omega_c$ is the transition frequency in the qubit between the excited $|+\rangle$ and the ground $|-\rangle$ levels, m is the photon multiple of transitions, and $|i\rangle_k$ are the virtual intermediate levels, the number of which is determined by the parameter $k = m - 1$.*

The interaction Hamiltonian of the model in the dipole and rotating-wave approximations is written as ($\hbar = 1$)

$$\hat{\mathcal{H}}_I = \gamma \sum_{i=A,B,C} \left(\hat{\sigma}_i^+ \hat{a}^m + \hat{\sigma}_i^- \hat{a}^{\dagger m} \right), \tag{1}$$

where $\hat{a}^\dagger$ ($\hat{a}$) is the creation (annihilation) operator of a cavity photon, $\hat{\sigma}_i^+ = |+\rangle_{ii}\langle -|$ and $\hat{\sigma}_i^- = |-\rangle_{ii}\langle +|$ are the raising and lowering operators for the i-th qubit. Index i in (1) numbers the qubits trapped in a cavity.

Let's choose the initial states of the system described by the three-qubit multiphoton Tavis-Cummings model (1). Suppose that the qubits A, B, and C at the initial moment of time $\tau \equiv \gamma t = 0$ are prepared in an arbitrary superposition of atomic states consisting of 8 ket vectors of Hilbert space:

$$\begin{aligned} |\Psi(0)\rangle_{ABC} &= \delta_1|+,+,+\rangle + \delta_2|+,+,-\rangle + \delta_3|+,-,+\rangle + \delta_4|-,+,+\rangle + \\ &+ \delta_5|+,-,-\rangle + \delta_6|-,+,-\rangle + \delta_7|-,-,+\rangle + \delta_8|-,-,-\rangle, \end{aligned} \tag{2}$$

and the field is in a vacuum state

$$|\Psi(0)\rangle_F = |0\rangle. \tag{3}$$

Here $\delta_i (i = 1, \ldots, 8)$ are coefficients that determine the degree of correlation between qubits A, B and C. Thus, we know the complete initial wave function of the system, which is given by the following expression: $|\Psi(0)\rangle_{ABCF} = |\Psi(0)\rangle_{ABC} \otimes |\Psi(0)\rangle_F$. For the system (1) under study and the initial conditions (2)–(3), we have exact solved the nonstationary Schrodinger equation $\mathrm{i}\frac{\partial|\Psi(\tau)\rangle_{ABCF}}{\partial\tau} = \hat{\mathcal{H}}_I|\Psi(\tau)\rangle_{ABCF}$ and calculated the total temporal density matrix of the "three qubits+resonator field mode"system $\hat{\wp}_{ABCF}(\tau) = |\Psi(\tau)\rangle_{ABCF\ ABCF}\langle\Psi(\tau)|$, which can be used to calculate any criteria for atomic correlations.

2. Selection of atomic correlation criteria and their modeling

To characterize quantum correlations between pairs of qubits, two criteria are used in this work: pairwise negativity $\xi_{ij}(\tau)$ and the symmetric quantum discord $Q^S_{ij}(\tau)$ ($i, j = A, B, C$; $i \neq j$). Negativity $\xi_{ij}(\tau)$ is defined in the standard way: $\xi_{ij}(\tau) = -2\sum_p \eta_p^{ij}(\tau)$, where the summation runs only over negative eigenvalues $\eta_p^{ij}(\tau)$ of the two-qubit density matrix $\hat{\wp}_{ij}^{\mathrm{T}_i}(\tau)$, on which the operation of partial transposition was performed over the variables of the i-th qubit.

To simplify further calculations and obtain analytical expressions for the criteria of quantum correlations, assume that the state of a pair of qubits i and j is described by a two-qubit density matrix $\hat{\wp}_{ij}(\tau)$, which has the X-shape ($i, j, k = A, B, C$; $i \neq j \neq k$):

$$\hat{\wp}_{ij}(\tau) = \mathrm{Tr}_F \mathrm{Tr}_k \hat{\wp}_{ABCF}(\tau) = \begin{pmatrix} \wp_{11}^{ij}(\tau) & 0 & 0 & \wp_{14}^{ij}(\tau) \\ 0 & \wp_{22}^{ij}(\tau) & \wp_{23}^{ij}(\tau) & 0 \\ 0 & (\wp_{23}^{ij}(\tau))^* & \wp_{33}^{ij}(\tau) & 0 \\ (\wp_{14}^{ij}(\tau))^* & 0 & 0 & \wp_{44}^{ij}(\tau) \end{pmatrix}. \tag{4}$$

Then, by performing the partial transpose operation T_i on the matrix (4), we can calculate its eigenvalues, two of which can be negative. Thus, the expression for negativity takes the form:
$\xi_{ij}(\tau) = -2\left[\eta_1^{ij}(\tau) + \eta_2^{ij}(\tau)\right]$, where $\eta_1^{ij} = \min\left[0, \frac{1}{2}\left(\wp_{11}^{ij} + \wp_{44}^{ij} - \sqrt{\left(\wp_{11}^{ij} - \wp_{44}^{ij}\right)^2 + 4|\wp_{23}^{ij}|^2}\right)\right]$,
$\eta_2^{ij} = \min\left[0, \frac{1}{2}\left(\wp_{22}^{ij} + \wp_{33}^{ij} - \sqrt{\left(\wp_{22}^{ij} - \wp_{33}^{ij}\right)^2 + 4|\wp_{14}^{ij}|^2}\right)\right]$.

The symmetric quantum discord $Q^S_{ij}(\tau)$ is defined as the average of the two one-way discords: $Q^S_{ij}(\tau) = (Q^R_{ij}(\tau) + Q^L_{ij}(\tau))/2$, where Q^R_{ij} (Q^L_{ij}) corresponds to a projective von Neumann measurement on qubit j (i). Each one-way discord is the difference between the total correlations $\mathcal{T}_{ij}(\tau)$ and the classical correlations $\mathcal{C}^R_{ij}(\tau)$ or $\mathcal{C}^L_{ij}(\tau)$: $Q^R_{ij} = \mathcal{T}_{ij} - \mathcal{C}^R_{ij}$, $Q^L_{ij} = \mathcal{T}_{ij} - \mathcal{C}^L_{ij}$. The total correlations are given by the quantum mutual information $\mathcal{T}_{ij} = S(\hat{\wp}_i) + S(\hat{\wp}_j) - S(\hat{\wp}_{ij})$, while the classical correlations are obtained by minimizing the conditional entropy over all projective measurements $\{\Pi_m^j\}$: $\mathcal{C}^R_{ij} = S(\hat{\wp}_i) - \min_{\{\Pi_m^j\}} S(\hat{\wp}_{i|j})$, $\mathcal{C}^L_{ij} = S(\hat{\wp}_j) - \min_{\{\Pi_m^i\}} S(\hat{\wp}_{j|i})$. Here $S(\hat{\wp}) = -\mathrm{Tr}(\hat{\wp}\log_2\hat{\wp})$ is the von Neumann entropy. For two-qubit X-states of the form (4), the minimization can be performed analytically: $\min_{\{\Pi_m^j\}} S(\hat{\wp}_{i|j}) = \min\left\{\mathcal{L}_1^{ij}, \mathcal{L}_2^{ij}\right\}$, $\min_{\{\Pi_m^i\}} S(\hat{\wp}_{j|i}) = \min\left\{\mathcal{G}_1^{ij}, \mathcal{G}_2^{ij}\right\}$, where the explicit expressions are $\mathcal{L}_1^{ij} = H\left(\frac{1+\sqrt{[1-2(\wp_{33}^{ij}+\wp_{44}^{ij})]^2+4(|\wp_{14}^{ij}|+|\wp_{23}^{ij}|)^2}}{2}\right)$, $\mathcal{L}_2^{ij} = -\sum_{l=1}^4 \wp_{ll}^{ij}\log_2\wp_{ll}^{ij} - H(\wp_{11}^{ij} + \wp_{33}^{ij})$, $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

As an example, Figures 2–3 demonstrate the dependence of symmetric quantum discord, symmetric classical and total correlations, and pairwise negativity on the reduced time τ for the initial state of qubits $|+,+,-\rangle$ with different multiplicities of photonic transitions m.

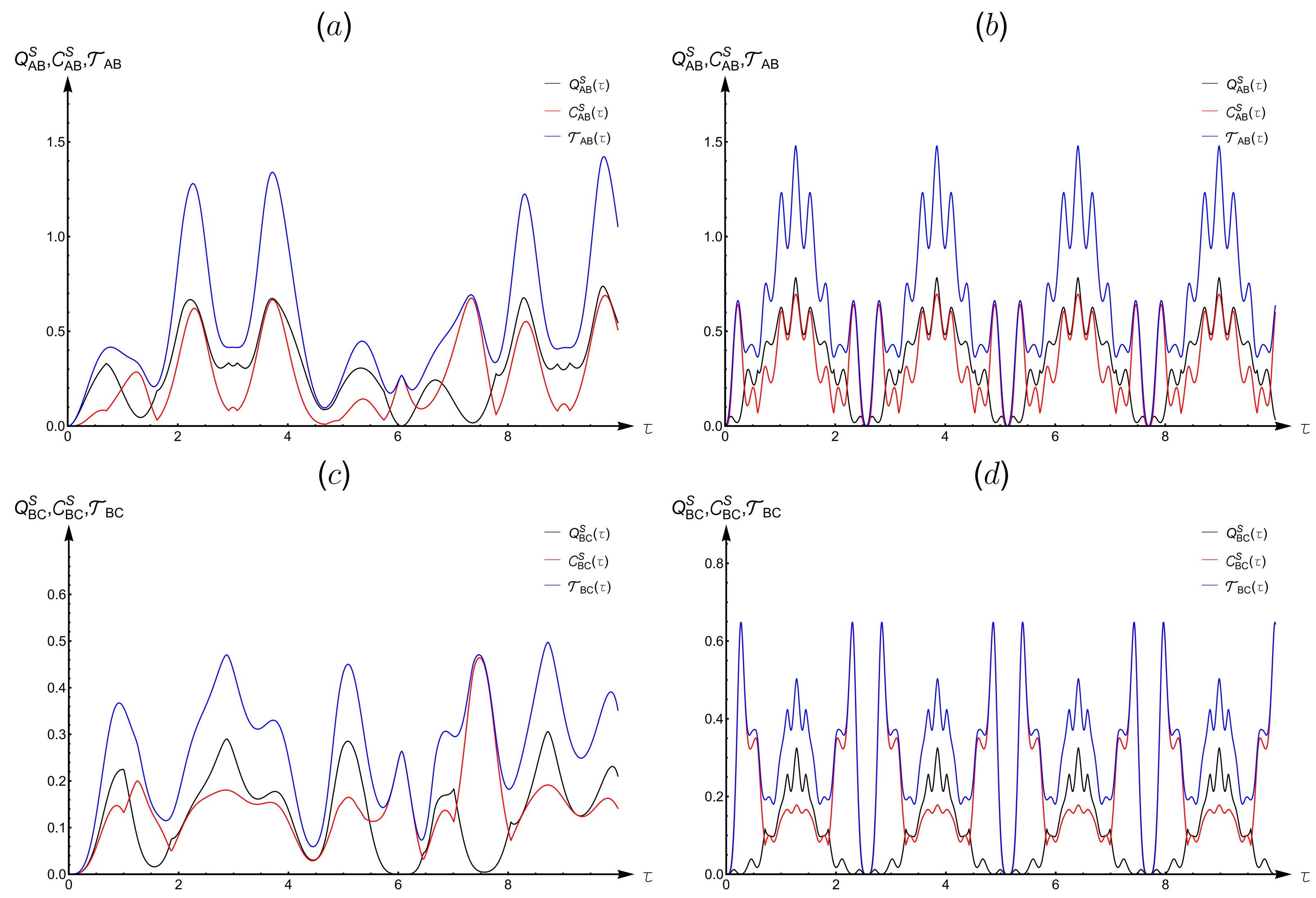

Figure 2: *Total correlations $\mathcal{T}_{ij}(\tau)$ (blue line), symmetric quantum discord $Q^S_{ij}(\tau)$ (black line), and symmetric classical correlations $\mathcal{C}^S_{ij}(\tau)$ (red line) are plotted as functions of the scaled time τ for the initial three-qubit state $|+,+,-\rangle$. The photon transition multiplicity is $m = 1$ for (a) and (c), and $m = 3$ for (b) and (d).*

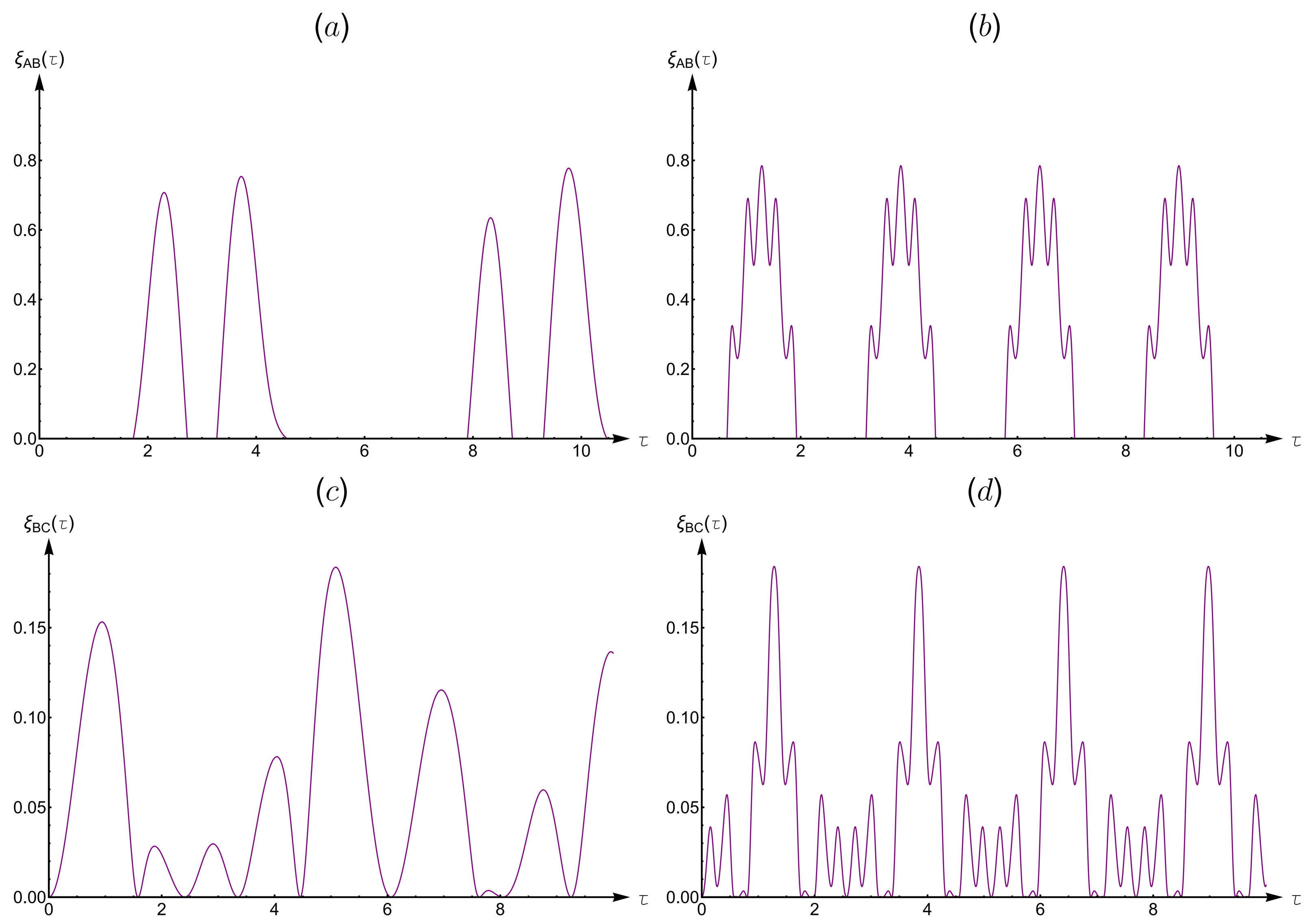

Figure 3: *Pairwise negativity $\xi_{ij}(\tau)$ are plotted as a function of the scaled time τ for the initial three-qubit state $|+,+,-\rangle$. The photon transition multiplicity is $m = 1$ for (a) and (c), and $m = 3$ for (b) and (d).*

The following conclusions can be drawn from the presented figures:

- For $|+,+,-\rangle$ increasing the multiplicity of photon transitions m does not suppress classical correlations, but even enhances them for a pair of qubits B and C. The dynamics of correlations and entanglement in three-photon processes ($m = 3$) has a clear periodic structure with periods of $2\pi/\sqrt{6}$ for zero values and $(2p+1)\pi/\sqrt{6}$ for maxima. At the moments of $\tau_p^{\min} = 2\pi p/\sqrt{6}$ the system returns to its initial separable state $|+,+,-\rangle$ and the vacuum state of the resonator field mode, and at time points $\tau_p^{\max} = (2p+1)\pi/\sqrt{6}$ – to the Werner type state. The negativity analysis for the pair A and B revealed the effects of sudden revival and death of entanglement with time intervals $\Delta_{rev} = 2(\pi - \sqrt{2})/\sqrt{6} \approx 1.41$ and $\Delta_{esd} = 2/\sqrt{3} \approx 1.155$; the area with entanglement exceeds the interval of its absence. In the regions of entanglement death, the symmetric quantum discord retains non-zero values due to significant mixing of the qubit subsystems.
- A similar analysis for the $|+,+,+\rangle$ state shows that both classical and quantum correlations with competitive dynamics arise in single-photon processes ($m = 1$). In three-photon processes ($m = 3$), classical correlations are suppressed, the symmetric quantum discord remains significant, and negativity becomes nonzero (entanglement occurs). At the same time, areas of sudden death of entanglement are observed, where the symmetric quantum discord retains positive values. This confirms that discord reveals a broader class of quantum correlations than negativity.